

# 8.3

复合求导与隐函数求导

同步练习

### 8.3.一

1. 设  $z = \frac{y}{x}$ , 而  $x = e^t$ ,  $y = 1 - \cos t$ , 则  $\frac{dz}{dt} = \underline{\hspace{2cm}}$

解 
$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = -\frac{y}{x^2} e^t + \frac{1}{x} \sin t \\ &= (\sin t + \cos t - 1)e^{-t}\end{aligned}$$

2. 设  $u = e^{x^2+y^2+z^2}$ , 而  $z = x^2 \sin y$ , 则  $\frac{\partial u}{\partial x} = \underline{\hspace{2cm}}$

解  $t = x^2 + y^2 + z^2, u = e^t$

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{du}{dt} \frac{\partial t}{\partial x} = e^{x^2+y^2+z^2} (2x + 2z \frac{\partial z}{\partial x}) \\ &= e^{x^2+y^2+z^2} (2x + 4xz \sin y) \\ &= 2xe^{x^2+y^2+z^2} (1 + 2z \sin y)\end{aligned}$$

### 8.3.一

3. 设方程  $\sin y + e^x = xy^2$  确定函数  $y(x)$ , 则  $\frac{dy}{dx} = \underline{\hspace{2cm}}$

解  $F(x, y) = \sin y + e^x - xy^2$      $F_x(x, y) = e^x - y^2$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$F_y(x, y) = \cos y - 2xy$$

$$= -\frac{e^x - y^2}{\cos y - 2xy}$$

$$\text{或 } -\cos y \frac{dy}{dx} + e^x = y^2 + 2xy \frac{dy}{dx}$$

$$\text{或 } \cos y dy + e^x dx = d(xy^2)$$

4. 设  $\frac{x}{z} = \ln \frac{z}{y}$ , 则  $\frac{\partial z}{\partial x} = \underline{\hspace{2cm}}$

解  $x = z \ln z - z \ln y$     设  $F = x - z \ln z + z \ln y$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{1}{-\ln z - 1 + \ln y}$$

$$F_x = 1, \quad F_y = \frac{z}{y},$$

$$= \frac{1}{\ln z + 1 - \ln y} = \frac{z}{x + z}$$

$$F_z = -\ln z - 1 + \ln y$$

### 8.3.二

1. 设  $z = \arctan(xy)$ ,  $y = e^x$ , 求  $\frac{dz}{dx}$ ,  $\frac{dz}{dy}$

解

$$\frac{dz}{dx} = \frac{1}{1 + (xy)^2} \left( y + x \frac{dy}{dx} \right) = \frac{1 + x}{1 + (xe^x)^2} e^x$$
$$\frac{dz}{dy} = \frac{1}{1 + (xy)^2} \left( y \frac{dx}{dy} + x \right) = \frac{1 + \ln y}{1 + (y \ln y)^2}$$

2. 设  $z = u^2 \ln v$ , 而  $u = \frac{x}{y}$ ,  $v = 4x - 3y$ , 求  $\frac{\partial z}{\partial x}$ ,  $\frac{\partial z}{\partial y}$

解

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = 2u \ln v \cdot \frac{1}{y} + \frac{u^2}{v} \cdot 4 \\ &= \frac{2u \ln v}{y} + \frac{4u^2}{v} \\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = 2u \ln v \left( -\frac{x}{y^2} \right) + \frac{u^2}{v} (-3) \\ &= -\frac{2ux \ln v}{y^2} - \frac{3u^2}{v} \end{aligned}$$

### 8.3.二

2. 设  $z = u^2 \ln v$ , 而  $u = \frac{x}{y}$ ,  $v = 4x - 3y$ , 求  $\frac{\partial z}{\partial x}$ ,  $\frac{\partial z}{\partial y}$

$$\begin{aligned}\text{解 } \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = 2u \ln v \cdot \frac{1}{y} + \frac{u^2}{v} \cdot 4 \\ &= \frac{2u \ln v}{y} + \frac{4u^2}{v} \\ &= \frac{2x \ln(4x - 3y)}{y^2} + \frac{4x^2}{(4x - 3y)y^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = 2u \ln v \left(-\frac{x}{y^2}\right) + \frac{u^2}{v} (-3) \\ &= -\frac{2ux \ln v}{y^2} - \frac{3u^2}{v} \\ &= -\frac{2x^3 \ln(4x - 3y)}{y^3} - \frac{3x^2}{(4x - 3y)y^2}\end{aligned}$$

### 8.3.二

3. 设 $z = f(x, u, v)$ ,  $u = 2x + y$ ,  $v = xy$ , 求复合函数 $z(x, y)$ 的全微分 $dz$

解

$$\begin{aligned} dz &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv \\ &= f'_1 dx + f'_2 du + f'_3 dv \\ &= f'_1 dx + f'_2 (2dx + dy) + f'_3 (ydx + xdy) \\ &= (f'_1 + 2f'_2 + yf'_3) dx + (f'_2 + xf'_3) dy \end{aligned}$$

### 8.3.二

4. 函数 $z = z(x, y)$ 由方程 $xz = \sin y + f(xy, z + y)$ 所确定, 其中 $f$ 具有一阶连续偏导, 求 $dz$ .

解

$$F(x, y, z) = \sin y + f(xy, z + y) - xz$$

$$F_x = f_1' y - z \quad F_y = \cos y + f_1' x + f_2' \quad F_z = f_2' - x$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{f_1' y - z}{f_2' - x}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\cos y + f_1' x + f_2'}{f_2' - x}$$

$$dz = \frac{1}{x - f_2'} [(yf_1' - z)dx + (\cos y + xf_1' + f_2')dy]$$

### 8.3.二

5. 设  $z = z(x, y)$  由  $z^5 - xz^4 + yz^3 = 1$  所确定, 求  $\left. \frac{\partial z}{\partial x} \right|_{\substack{x=0 \\ y=0}}, \left. \frac{\partial z}{\partial y} \right|_{\substack{x=0 \\ y=0}}$

解 设  $F(x, y) = z^5 - xz^4 + yz^3 - 1$

$$F_x = -z^4 \quad F_y = z^3 \quad F_z = 5z^4 - 4xz^3 + 3yz^2$$

由  $z^5 - xz^4 + yz^3 = 1, x = 0, y = 0$ , 得  $z^5 = 1, z = 1$

$$\left. \frac{\partial z}{\partial x} \right|_{\substack{x=0 \\ y=0}} = - \left. \frac{F_x}{F_z} \right|_{\substack{x=0 \\ y=0}} = \left. \frac{z^4}{5z^4 - 4xz^3 + 3yz^2} \right|_{\substack{x=0 \\ y=0}} = \frac{1}{5}$$

$$\left. \frac{\partial z}{\partial y} \right|_{\substack{x=0 \\ y=0}} = \left. \frac{F_y}{F_z} \right|_{\substack{x=0 \\ y=0}} = \left. \frac{z^3}{5z^4 - 4xz^3 + 3yz^2} \right|_{\substack{x=0 \\ y=0}} = \frac{1}{5}$$

### 8.3.二

6. 设  $u = f(x, y, z)$  具有一阶连续偏导,  $y = y(x), z = z(x)$  由方程  $e^{xy} - xy = 2$  及  $e^x = \int_0^{x-z} \frac{\sin t}{t} dt$  确定, 求  $\frac{du}{dx}$ .

$$\begin{aligned}\text{解} \quad du &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = f'_1 dx + f'_2 dy + f'_3 dz \\ &= \left\{ f'_1 - \frac{y}{x} f'_2 - \frac{1}{\sin(x-z)} [(x-z)e^x - \sin(x-z)] f'_3 \right\} dx \\ \frac{du}{dx} &= f'_1 - \frac{y}{x} f'_2 - \frac{1}{\sin(x-z)} [(x-z)e^x - \sin(x-z)] f'_3\end{aligned}$$

$$\text{而 } e^{xy}(ydx + xdy) - (ydx + xdy) = 0 \quad dy = -\frac{y}{x} dx$$

$$e^x dx = \frac{\sin(x-z)}{x-z} (dx - dz)$$

$$dz = -\frac{1}{\sin(x-z)} [(x-z)e^x - \sin(x-z)] dx$$

### 8.3.二

7. 设  $z = x^3 f(xy, \frac{y}{x})$ , 而  $f(u, v)$  有连续二阶偏导数, 求  $\frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$ .

$$\text{解 } \frac{\partial z}{\partial y} = x^3 (x f'_1 + \frac{1}{x} f'_2) = x^4 f'_1 + x^2 f'_2$$

$$\frac{\partial^2 z}{\partial y^2} = x^5 f''_{11} + x^3 f''_{12} + x^3 f''_{21} + x f''_{22} = x^5 f''_{11} + 2x^3 f''_{12} + x f''_{22}$$

$$\frac{\partial z}{\partial x} = 3x^2 f + x^3 (y f'_1 - \frac{y}{x^2} f'_2) = 3x^2 f + x^3 y f'_1 - x y f'_2$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= 3x^2 (x f'_1 + \frac{1}{x} f'_2) + x^3 f'_1 + x^3 y (x f''_{11} + \frac{1}{x} f''_{12}) - x y f'_2 \\ &\quad - x y (\cancel{x f''_{21}} + \frac{1}{x} f''_{22}) \\ &= 4x^3 f'_1 - 2x^2 f'_2 + x^4 y f''_{11} - y f''_{22} \end{aligned}$$

### 8.3.二

8. 设方程组  $\begin{cases} z = xf(x+y) \\ F(x, y, z) = 0 \end{cases}$  确定  $y(x), z(x)$ , 求  $\frac{dz}{dx}$

解  $\begin{cases} xf(x+y) - z = 0 \\ F(x, y, z) = 0 \end{cases} \quad \begin{cases} G = xf(x+y) - z \\ F = F(x, y, z) \end{cases}$

雅可比矩阵  $\begin{pmatrix} G_x & G_y & G_z \\ F_x & F_y & F_z \end{pmatrix} = \begin{pmatrix} f + xf' & xf' & -1 \\ F_x & F_y & F_z \end{pmatrix}$

$$\begin{aligned} \frac{dz}{dx} &= - \frac{\begin{vmatrix} G_y & G_x \\ F_y & F_x \end{vmatrix}}{\begin{vmatrix} G_y & G_z \\ F_y & F_z \end{vmatrix}} = - \frac{\begin{vmatrix} xf' & f + xf' \\ F_y & F_x \end{vmatrix}}{\begin{vmatrix} xf' & -1 \\ F_y & F_z \end{vmatrix}} = - \frac{xf'F_x - (f + xf')F_y}{xf'F_z + F_y} \\ &= \frac{(f + xf')F_y - xf'F_x}{xf'F_z + F_y} \end{aligned}$$

### 8.3.二

8. 设方程组  $\begin{cases} z = xf(x+y) \\ F(x, y, z) = 0 \end{cases}$  确定  $y(x), z(x)$ , 求  $\frac{dz}{dx}$

$$\text{解 } \begin{cases} xf(x+y) - z = 0 \\ F(x, y, z) = 0 \end{cases} \quad \begin{cases} f + xf' + \frac{dy}{dx} xf' - \frac{dz}{dx} = 0 \\ F_x + F_y \frac{dy}{dx} + F_z \frac{dz}{dx} = 0 \end{cases}$$

$$\begin{cases} F_y f + xf' F_y + \frac{dy}{dx} xf' F_y - \frac{dz}{dx} F_y = 0 \\ xf' F_x + F_y \frac{dy}{dx} xf' + xf' F_z \frac{dz}{dx} = 0 \end{cases}$$

$$F_y f + xf' F_y - \frac{dz}{dx} F_y - xf' F_x - xf' F_z \frac{dz}{dx} = 0$$

$$\frac{dz}{dx} = \frac{F_y f + xf' F_y - xf' F_x}{xf' F_z + F_y} = \frac{(f + xf') F_y - xf' F_x}{xf' F_z + F_y}$$

### 8.3.二

9. 设方程组  $\begin{cases} xu - yv = 0 \\ yu + xv = 1 \end{cases}$  确定  $u(x, y), v(x, y)$ ,

求  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ .

雅可比矩阵

解  $\begin{cases} F = xu - yv \\ G = yu + xv - 1 \end{cases} \quad \begin{pmatrix} F_x & F_y & F_u & F_v \\ G_x & G_y & G_u & G_v \end{pmatrix} = \begin{pmatrix} u & -v & x & -y \\ v & u & y & x \end{pmatrix}$

$$\frac{\partial u}{\partial x} = - \frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = - \frac{\begin{vmatrix} u & -y \\ v & x \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} = - \frac{ux + yv}{x^2 + y^2}$$

$$\frac{\partial v}{\partial y} = - \frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = - \frac{\begin{vmatrix} x & -v \\ y & u \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} = - \frac{xu + yv}{x^2 + y^2}$$

### 8.3.二

9. 设方程组  $\begin{cases} xu - yv = 0 \\ yu + xv = 1 \end{cases}$  确定  $u(x, y), v(x, y)$ ,

求  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ .

雅可比矩阵

解  $\begin{cases} F = xu - yv \\ G = yu + xv - 1 \end{cases} \quad \begin{pmatrix} F_x & F_y & F_u & F_v \\ G_x & G_y & G_u & G_v \end{pmatrix} = \begin{pmatrix} u & -v & x & -y \\ v & u & y & x \end{pmatrix}$

$$\frac{\partial u}{\partial y} = - \frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = - \frac{\begin{vmatrix} -v & -y \\ u & x \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} = - \frac{yu - xv}{x^2 + y^2}$$

$$\frac{\partial v}{\partial x} = - \frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = - \frac{\begin{vmatrix} x & u \\ y & v \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} = - \frac{xv - yu}{x^2 + y^2}$$

### 8.3.三

1. 设函数 $z(x, y)$ 由 $F(x - y, y - z, z - x) = 0$ 确定,  $F(u, v, w)$

具有连续偏导, 且 $F'_v - F'_w \neq 0$ , 证明 $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$ .

证一: 
$$F'_1 - F'_2 \frac{\partial z}{\partial x} + F'_3 \left( \frac{\partial z}{\partial x} - 1 \right) = 0$$

$$\frac{\partial z}{\partial x} = \frac{F'_1 - F'_3}{F'_2 - F'_3}$$

$$-F'_1 + F'_2 \left( 1 - \frac{\partial z}{\partial y} \right) + F'_3 \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{F'_2 - F'_1}{F'_2 - F'_3}$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{F'_3 - F'_1}{F'_2 - F'_3} + \frac{F'_2 - F'_1}{F'_2 - F'_3} = 1$$

### 8.3.三

1. 设函数 $z(x, y)$ 由 $F(x - y, y - z, z - x) = 0$ 确定,  $F(u, v, w)$ 具有连续偏导, 且 $F'_v - F'_w \neq 0$ , 证明 $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$ .

证二: 令  $G(x, y, z) = F(x - y, y - z, z - x)$

$$G_x = F'_1 \cdot 1 + F'_3 \cdot (-1) \quad G_y = F'_1 \cdot (-1) + F'_2 \cdot 1$$

$$G_z = F'_2 \cdot (-1) + F'_3 \cdot 1$$

$$\frac{\partial z}{\partial x} = -\frac{G_x}{G_z} = \frac{F'_1 - F'_3}{F'_2 - F'_3} \quad \frac{\partial z}{\partial y} = -\frac{G_y}{G_z} = \frac{F'_2 - F'_1}{F'_2 - F'_3}$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{F'_3 - F'_1}{F'_2 - F'_3} + \frac{F'_2 - F'_1}{F'_2 - F'_3} = 1$$

### 8.3.三

2. 设函数 $z(x, y)$ 由 $\phi(x + \frac{z}{y}, y + \frac{z}{x}) = 0$ 确定,  $\phi(u, v)$

具有连续偏导, 证明 $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z - xy$ .

$$\text{证一: } \varphi_1(1 + \frac{1}{y} \frac{\partial z}{\partial x}) + \varphi_2[\frac{1}{x^2}(x \frac{\partial z}{\partial x} - z)] = 0$$

$$\frac{\partial z}{\partial x} = (\frac{z}{x^2} \varphi_2 - \varphi_1) / (\frac{1}{x} \varphi_2 + \frac{1}{y} \varphi_1)$$

$$\varphi_1[\frac{1}{y^2}(y \frac{\partial z}{\partial y} - z)] + \varphi_2(1 + \frac{1}{x} \frac{\partial z}{\partial y}) = 0$$

$$\frac{\partial z}{\partial y} = (\frac{z}{y^2} \varphi_1 - \varphi_2) / (\frac{1}{x} \varphi_2 + \frac{1}{y} \varphi_1) = z - xy.$$

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{1}{\frac{1}{x} \varphi_2 + \frac{1}{y} \varphi_1} [x(\frac{z}{x^2} \varphi_2 - \varphi_1) + y(\frac{z}{y^2} \varphi_1 - \varphi_2)] = z - xy.$$

### 8.3.三

2. 设函数 $z(x, y)$ 由 $\phi(x + \frac{z}{y}, y + \frac{z}{x}) = 0$ 确定,  $\phi(u, v)$

具有连续偏导, 证明 $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z - xy$ .

证二:  $F(x, y) = \phi(x + \frac{z}{y}, y + \frac{z}{x})$

$$F_x = \phi_1 \cdot 1 + \phi_2 \cdot z(-\frac{1}{x^2}) \quad F_y = \phi_1 \cdot (-\frac{1}{y^2}) + \phi_2 \cdot 1$$

$$F_z = \phi_1 \cdot \frac{1}{y} + \phi_2 \cdot \frac{1}{x}$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = (\frac{z}{x^2} \phi_2 - \phi_1) / (\frac{1}{x} \phi_2 + \frac{1}{y} \phi_1)$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = (\frac{z}{y^2} \phi_1 - \phi_2) / (\frac{1}{x} \phi_2 + \frac{1}{y} \phi_1) \quad \text{代入得证}$$